

# Lagrangian Perturbation Approach to the Formation of Large-scale Structure\*

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**Abstract.** The present lecture notes address three columns on which the Lagrangian perturbation approach to cosmological dynamics is based: 1. the formulation of a Lagrangian theory of self-gravitating flows in which the dynamics is described in terms of a single field variable; 2. the procedure, how to obtain the dynamics of Eulerian fields from the Lagrangian picture, and 3. a precise definition of a Newtonian cosmology framework in which Lagrangian perturbation solutions can be studied. While the first is a discussion of the basic equations obtained by transforming the Eulerian evolution and field equations to the Lagrangian picture, the second exemplifies how the Lagrangian theory determines the evolution of Eulerian fields including kinematical variables like expansion, vorticity, as well as the shear and tidal tensors. The third column is based on a specification of initial and boundary conditions, and in particular on the identification of the average flow of an inhomogeneous cosmology with a “Hubble-flow”. Here, we also look at the limits of the Lagrangian perturbation approach as inferred from comparisons with N-body simulations and illustrate some striking properties of the solutions.

\* to appear in: Proc. Int. School of Physics Enrico Fermi, Course CXXXII, Varenna 1995.

## 1. Lagrangian Theory of Self-gravitating Flows

The description of fluid motions in cosmology has been largely studied in an *Eulerian* coordinate system  $\vec{x}$ , i.e., a rectangular non-rotating frame in Euclidean space. Quite recently, it has become popular to study fluid motions in a *Lagrangian* coordinate system  $\vec{X}$ , i.e., a curvilinear, possibly rotating frame in Euclidean space which is defined such as to move with the fluid. Since the Lagrangian description has a number of advantages over the Eulerian one, and since this description enjoys many applications in the recent cosmology literature, it is important to elucidate in proper language the Lagrangian formalism. Since the lectures by François Bouchet and Peter Coles (this volume) explore the field of recent applications of the Lagrangian perturbation theory, I here concentrate on the basic architecture of a Lagrangian theory of structure formation. I do this in Newtonian cosmology, the lecture by Sabino Matarrese (this volume) gives an extension

to the framework of General Relativity. Accordingly, I kept my reference list short, since more references may be found in the other lectures.

That the Lagrangian approach is experiencing a revival in cosmology is good news; I consider it the natural frame to describe fluid motions, since this description is formally close to the mechanics of point particles. If you consult old textbooks on hydrodynamics, you will find that the Lagrangian picture was considered too complicated for practical purposes beyond problems with high symmetry, and therefore has not been pursued further. I hope that, after this and the related lectures, you will be convinced of the opposite.

Let us start with the basic system of equations in Newtonian cosmology describing the motion of a pressureless fluid in the gravitational field which is generated by its own density. We think at applications for a (dominating) collisionless component in the Universe; the gravitational dynamics we describe is thought to act as an attractor for the baryonic matter component which is “lighted up” by physics not described by these equations.

With this assumption the fluid motion in Eulerian space is completely characterized by its velocity field  $\vec{v}(\vec{x}, t)$  and its density field  $\varrho(\vec{x}, t) > 0$ . The fluid has to obey the familiar *evolution equations* for these fields,

$$\partial_t \vec{v} = -(\vec{v} \cdot \nabla) \vec{v} + \vec{g} , \quad (1a)$$

$$\partial_t \varrho = -\nabla \cdot (\varrho \vec{v}) , \quad (1b)$$

where the gravitational field  $\vec{g}(\vec{x}, t)$  is constrained by the (Newtonian) *field equations*

$$\nabla \times \vec{g} = \vec{0} , \quad (1c)$$

$$\nabla \cdot \vec{g} = \Lambda - 4\pi G \varrho ; \quad (1d)$$

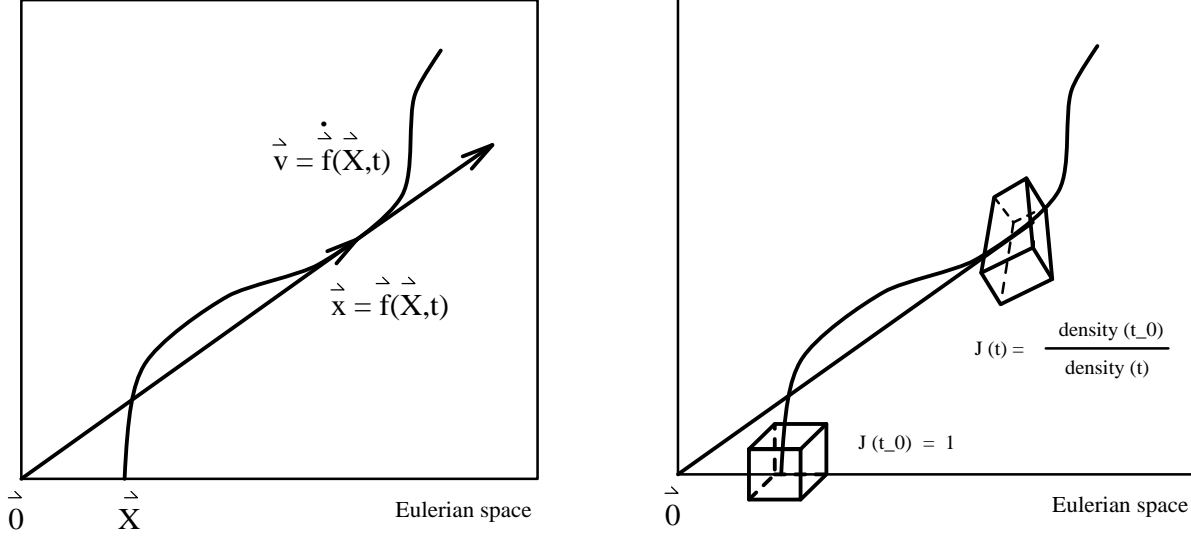
$\Lambda$  denotes the cosmological constant. (Strictly speaking,  $\vec{g}$  in eq. (1a) is a force per unit *inertial mass*, whereas in eqs. (1c,d)  $\vec{g}$  is the field strength associated with *gravitational mass*. That we set both equal is the content of Einstein’s equivalence principle of inertial and gravitational mass.)

Alternatively, we can write the eqs. (1c) and (1d) in terms of a single Poisson equation for the gravitational potential, which we do not need in the following. Hereafter, we call the system (1) the *Euler–Newton system*.

One important issue to learn about the Lagrangian treatment is the fact that both evolution equations (1a) and (1b) can be integrated exactly in Lagrangian space, velocity and density will therefore not appear as dynamical variables later. To see this, we first look at the basic Lagrangian field variable which is the *trajectory field* of fluid elements, or the *deformation field* of the medium, respectively (Fig.1a):

$$\vec{x} = \vec{f}(\vec{X}, t) ; \quad \vec{X} := \vec{f}(\vec{X}, t_0) , \quad (2a)$$

where  $\vec{X}$  denote the Lagrangian coordinates which label fluid elements,  $\vec{x}$  are the positions of these elements in Eulerian space at the time  $t$ , and  $\vec{f}$  is the trajectory of fluid elements for constant  $\vec{X}$ . (Notice: the Eulerian positions  $\vec{x}$  are here viewed not as independent variables (i.e., coordinates), but as dependent fields of Lagrangian coordinates; therefore, we employ the letter  $\vec{f}$  for the sake of clarity. Independent variables are now  $(\vec{X}, t)$  instead of  $(\vec{x}, t)$ .)



**Figure 1.** The Newtonian spacetime projected onto an Eulerian plane is sketched. A fluid element sitting initially ( $t = t_0$ ) at  $\vec{X}$  moves along the trajectory  $\vec{f}$  to the Eulerian position  $\vec{x}$  at time  $t$ ; this position vector is expressed in terms of Lagrangian (i.e. initial) coordinates which are constant along the trajectory:  $\vec{X} = \vec{0}$  (Fig.1a). The Jacobian  $J$  measures the volume deformation of fluid elements located at  $\vec{X}$ ;  $J$  drops from 1 to zero as the volume element degenerates into a surface element (piece of pancake), a line element (piece of filament), or a point (cluster).

The velocity field  $\vec{v}(\vec{x}, t)$  of the fluid is the tangential field to this family of curves,

$$\vec{v} = \dot{\vec{f}}(\vec{X}, t) , \quad (2b)$$

and the dot denotes the total (Lagrangian or convective) time-derivative along  $\vec{v}$ :

$$(\dots)^{\cdot} \equiv \frac{d}{dt} := \partial_t + \vec{v} \cdot \nabla . \quad (2c)$$

(The dot commutes with Lagrangian differentiation.)

In view of (2c) we recognize that the Eulerian evolution equation (1a) can be written as  $\dot{\vec{v}} = \vec{g}$ , which is the familiar definition of acceleration (here, the acceleration of a fluid element along its trajectory). Consequently, in view of (2b), we know the acceleration

field in Lagrangian space for *any given* trajectory:

$$\vec{g} = \ddot{\vec{f}}(\vec{X}, t) . \quad (2d)$$

In other words, taking (2b) and (2d) as *definitions* of velocity and acceleration, respectively (like in point mechanics!), we see that equation (1a) is automatically fulfilled as we switch to the Lagrangian description.

A similar logic applies to the continuity equation (1b): the deformation of the medium is described by the Lagrangian *deformation tensor*  $(f_{i|k})$ , i.e., the tensor of first derivatives of the trajectory field with respect to Lagrangian coordinates  $\dagger$  which measures how much the Eulerian positions deviate from their original (Lagrangian) positions. The volume of the deformed fluid element is measured by the determinant of this tensor,  $J(\vec{X}, t) := \det(f_{i|k})$ , and therefore its density must be inversely proportional to it (Fig.1b):

$$\varrho(\vec{X}, t) = \varrho J^{-1} . \quad (2e)$$

( $\varrho := \varrho(\vec{X}, t_0)$ ) is the initial density field, and  $J(\vec{X}, t_0) = 1$  according to our definition of the Lagrangian coordinates in (2a) which coincide with the Eulerian ones at  $t = t_0$ .) You may verify (2e) by differentiation and by using the identity

$$\dot{J} = J \nabla \cdot \vec{v} . \quad (2f)$$

Again we conclude that, for *any given*  $\vec{f}(\vec{X}, t)$ , we obtain the exact expression for the density at the fluid element, i.e., in Lagrangian space.

Let us now move to the question which *Lagrangian evolution equations* the field of trajectories has to obey. We obtain them by expressing the field equations (1c) and (1d) (which are 4 linear “constraint equations” for the acceleration field) in terms of Lagrangian coordinates which, as we shall see, yields 4 non-linear partial differential equations for  $\vec{g}(\vec{X}, t)$ . For this purpose we need the inverse of the transformation (2a) from Lagrangian to Eulerian coordinates,

$$\vec{X} = \vec{h}(\vec{x}, t) ; \quad \vec{h} \equiv \vec{f}^{-1} . \quad (2g)$$

While the deformation tensor is the Jacobian matrix of the transformation from Eulerian to Lagrangian coordinates,  $f_{i|k} = J_{ik}$ , the inverse of  $f_{i|k}$  is the inverse Jacobian matrix  $\ddagger$

$$h_{a,b} = J_{ab}^{-1} = \text{ad}(J_{ab})J^{-1} = \frac{1}{2J} \epsilon_{ajk} \epsilon_{b\ell m} f_{\ell|j} f_{m|k} . \quad (2h)$$

$\dagger$  Lagrangian differentiation is indicated throughout this lecture with a vertical slash to make a difference to differentiation with respect to Eulerian coordinates, denoted by commata.

$\ddagger$  Hereafter we adopt the summation convention.

Since the transformation part “looks” inconvenient during a first reading of papers on Lagrangian models, I here try to be as elementary as possible.

We look at the two-dimensional case first, i.e., there are only two non-vanishing field components, e.g.,  $g_1$  and  $g_2$  <sup>†</sup>. We then get for the Eulerian derivatives of  $\vec{g}$ :

$$\begin{aligned} g_{1,1} &= g_{1|1}h_{1,1} + g_{1|2}h_{2,1} \quad , \quad g_{1,2} = g_{1|1}h_{1,2} + g_{1|2}h_{2,2} \quad , \\ g_{2,1} &= g_{2|1}h_{1,1} + g_{2|2}h_{2,1} \quad , \quad g_{2,2} = g_{2|1}h_{1,2} + g_{2|2}h_{2,2} \quad . \end{aligned} \quad (3a)$$

Inserting the inverse Jacobian matrix,

$$J_{ab}^{-1}(2D) = \frac{1}{J} \begin{pmatrix} f_{2|2} & -f_{1|2} \\ -f_{2|1} & f_{1|1} \end{pmatrix} \quad , \quad (3b)$$

with  $J(2D) = f_{1|1}f_{2|2} - f_{1|2}f_{2|1}$ , we can express the single component of the curl and the divergence of  $\vec{g}$  in terms of Lagrangian derivatives of  $\vec{g}(\vec{X}, t)$  and  $\vec{f}(\vec{X}, t)$ :

$$g_{2,1} - g_{1,2} = (g_{2|1}f_{2|2} - g_{2|2}f_{2|1} + g_{1|1}f_{1|2} - g_{1|2}f_{1|1}) J^{-1} \quad , \quad (4a)$$

$$g_{1,1} + g_{2,2} = (g_{1|1}f_{2|2} - g_{1|2}f_{2|1} - g_{2|1}f_{1|2} + g_{2|2}f_{1|1}) J^{-1} \quad . \quad (4b)$$

Using the exact integrals (2d) and (2e) of eqs. (1a) and (1b), eqs. (1c) and (1d) are transformed into a closed Newtonian system for the trajectories in 2D:

$$\ddot{f}_{2|1}f_{2|2} - \ddot{f}_{2|2}f_{2|1} + \ddot{f}_{1|1}f_{1|2} - \ddot{f}_{1|2}f_{1|1} = 0 \quad , \quad (5a)$$

$$\ddot{f}_{1|1}f_{2|2} - \ddot{f}_{1|2}f_{2|1} - \ddot{f}_{2|1}f_{1|2} + \ddot{f}_{2|2}f_{1|1} = \Lambda J(2D) - 4\pi G \varrho^0 \quad . \quad (5b)$$

(Notice that we have multiplied with  $J$ , i.e., we must ensure that  $J \neq 0$ .) Looking at the equations (5) we see that we have just written out functional determinants:

$$\frac{\partial(\ddot{f}_1, f_1)}{\partial(X_1, X_2)} + \frac{\partial(\ddot{f}_2, f_2)}{\partial(X_1, X_2)} = 0 \quad , \quad (5a)$$

$$\frac{\partial(\ddot{f}_1, f_2)}{\partial(X_1, X_2)} - \frac{\partial(\ddot{f}_2, f_1)}{\partial(X_1, X_2)} = \Lambda \frac{\partial(f_1, f_2)}{\partial(X_1, X_2)} - 4\pi G \varrho^0 \quad . \quad (5b)$$

In 3D we have to employ some more tensor algebra, but the procedure is the same. The relevant formula for the transformation of any tensor (here exemplified for the acceleration gradient  $(g_{i,j})$ ) reads:

$$g_{i,j} = g_{i|k}h_{k,j} = \frac{1}{2J} \epsilon_{k\ell m} \epsilon_{j p q} g_{i|k} f_{p|\ell} f_{q|m} \quad , \quad (6)$$

<sup>†</sup> We are talking about a purely two-dimensional space and not about cylinders in the third direction in which case we would have  $f_3 = X_3$ .

where we have used the formula for the inverse Jacobian (2h). In view of the definition of a functional determinant of any three functions  $A(\vec{X}, t)$ ,  $B(\vec{X}, t)$  and  $C(\vec{X}, t)$ ,

$$\mathcal{J}(A, B, C) := \frac{\partial(A, B, C)}{\partial(X_1, X_2, X_3)} = \epsilon_{k\ell m} A_{|k} B_{|\ell} C_{|m} \ ,$$

e.g., for the Jacobian determinant we simply have  $J = \mathcal{J}(f_1, f_2, f_3)$ , we can write equation (6) as

$$g_{i,j} = \frac{1}{2J} \epsilon_{jpq} \mathcal{J}(g_i, f_p, f_q) \ . \quad (6)$$

The curl and the divergence of the acceleration field can be read from eq. (6) as the anti-symmetric part of the acceleration gradient and its trace (here, repeated indices imply summation as before, but with  $i, j, k$  running through the cyclic permutations of 1, 2, 3):

$$g_{[i,j]} = -\frac{1}{2}(\nabla \times \vec{g})_k = \frac{1}{2} \epsilon_{pq[j} \mathcal{J}(\ddot{f}_i, f_p, f_q) J^{-1} \ , \quad (7a,b,c)$$

$$g_{i,i} = (\nabla \cdot \vec{g}) = \frac{1}{2} \epsilon_{abc} \mathcal{J}(\ddot{f}_a, f_b, f_c) J^{-1} \ . \quad (7d)$$

Inserting (7) and the exact integrals (2d) and (2e) into (1c) and (1d) we finally obtain the *Lagrange–Newton system* [7](no background source, in particular  $\Lambda = 0$ ) and [8](including backgrounds of Friedmann type):

$$\mathcal{J}(\ddot{f}_1, f_1, f_3) + \mathcal{J}(\ddot{f}_2, f_2, f_3) = 0 \ , \quad (8a)$$

$$\mathcal{J}(\ddot{f}_2, f_2, f_1) + \mathcal{J}(\ddot{f}_3, f_3, f_1) = 0 \ , \quad (8b)$$

$$\mathcal{J}(\ddot{f}_1, f_1, f_2) + \mathcal{J}(\ddot{f}_3, f_3, f_2) = 0 \ , \quad (8c)$$

$$\begin{aligned} \mathcal{J}(\ddot{f}_1, f_2, f_3) + \mathcal{J}(\ddot{f}_2, f_3, f_1) + \mathcal{J}(\ddot{f}_3, f_1, f_2) \\ - \Lambda \mathcal{J}(f_1, f_2, f_3) = -4\pi G \varrho \ . \end{aligned} \quad (8d)$$

I have written this system here in explicit form to make working with these equations most convenient. (Alternative forms of these equations may be found in [21]; in that paper we also employ the calculus of differential forms, which makes the above derivation even simpler.)

The Lagrange–Newton system (8) is the basic system of equations we want to study. Although these equations look, and in fact are more complicated than their Eulerian counterparts, they have proven to be as useful for finding exact solutions as well as perturbative approximations beyond the linear regime. Here, I note that the rules of determinant manipulations apply to these equations, and we may evaluate many problems analytically.

Let us summarize the main conclusions of this section:

- The Eulerian evolution equations for the velocity and density fields (1a) and (1b) are integrated exactly in the Lagrangian picture by (2d) and (2e), respectively.
- The transformation of the Eulerian field equations (1c,d) yields a system of Lagrangian field equations for the acceleration field.
- Using the integrals for the acceleration and density fields (2d) and (2e) we arrive at the Lagrange–Newton system (8) which is a closed system of Lagrangian evolution equations for the field of trajectories. Density and velocity are no longer dynamical variables, but are replaced by the single dynamical field  $\vec{f}(\vec{X}, t)$ .
- The Lagrange–Newton system (8) is equivalent to the Euler–Newton system (1) as long as the mapping  $\vec{f}$  is invertible ( $J > 0$ ) and, in particular, non-singular ( $J \neq 0$ ); (for a proof see [21]). Note that the Lagrange–Newton system remains regular at caustics ( $J = 0$ ) where the Eulerian representation breaks down. Whether the Lagrangian equations still describe the physics correctly in the regime  $J < 0$  will be discussed in Section 3.

## 2. Lagrangian Dynamics of Eulerian Fields

The question how to obtain the evolution of Eulerian fields from the Lagrangian description is easily answered: Knowing the Lagrangian solution of the transformation (2a),  $\vec{x} = \vec{f}(\vec{X}, t)$ , we have to express the Eulerian fields first in terms of  $\vec{f}$ , and then use the inverse of this transformation (2g),  $\vec{X} = \vec{h}(\vec{x}, t)$ , to map the variable  $\vec{X}$  back to Eulerian space, e.g., for the velocity we have:

$$\vec{v}(\vec{x}, t) = \dot{\vec{f}}(\vec{h}(\vec{x}, t), t) . \quad (9)$$

The power of a Lagrangian description mainly relies on this implicit determination of the evolution of Eulerian fields. However, we can only come back to Eulerian space as long as  $\vec{h}$  exists. In general, the inverse transformation can be multivalued (see the discussion in Section 3).

The same rule applies to *any* Eulerian field, so we only have to find the corresponding formula which expresses it in terms of  $\vec{f}$ .

Let us now discuss another way of describing the evolution of Eulerian fields along trajectories. This will lead us to equations which are frequently discussed in the literature. These equations are useful to illuminate the power of the Lagrangian

formalism outlined in Section 1, but they do not determine the evolution of Eulerian fields *per se*, as will be shown below.

We may ask for evolution equations which involve the Lagrangian time-derivative (2c) instead of the Eulerian time-derivative  $\partial_t$  as in (1a) and (1b). They may be written in symbolic form as:

$$\dot{\mathcal{E}}_\nu = \mathcal{F}_\nu(\mathcal{E}_\mu) \ ,$$

i.e., a system of  $\nu$  equations which determines the evolution of the  $\nu$  variables  $\mathcal{E}_\nu$  along the flow lines. A solution of this system of equations does not provide the full answer, which needs knowledge of the trajectories themselves! We will see that the procedure outlined at the beginning of this subsection *does provide* the full answer. We shall derive now *Lagrangian evolution equations* for a number of fields of interest, and shall then reinforce the Lagrange-Newton system (8).

We start with eq. (1a) and write it down in index notation,

$$\partial_t v_i + v_k v_{i,k} = g_i \ . \quad (10a)$$

Performing the spatial Eulerian derivative of this equation and using (2c) we obtain:

$$(v_{i,j})^\bullet = -v_{i,k} v_{k,j} + g_{i,j} \ . \quad (10b)$$

Eq. (10b) is an evolution equation for the velocity gradient  $(v_{i,j})$  along the flow lines  $\vec{f}$ . It is convenient for the discussion of fluid motions to split it into its symmetric part (expansion tensor  $\theta_{ij}$ ), its antisymmetric part (vorticity tensor  $\omega_{ij}$ ), and to separate the symmetric part into a tracefree part (the shear tensor  $\sigma_{ij}$ ) and the trace (the rate of expansion)  $\theta := v_{i,i}$ :

$$v_{i,j} = v_{(i,j)} + v_{[i,j]} =: \theta_{ij} + \omega_{ij} =: \sigma_{ij} + \frac{1}{3}\theta\delta_{ij} + \omega_{ij} \ , \quad (11)$$

where  $v_{(i,j)} = \frac{1}{2}(v_{i,j} + v_{j,i})$  and  $v_{[i,j]} = \frac{1}{2}(v_{i,j} - v_{j,i})$ .

Inserting (11) into (10b) we obtain the following evolution equations (compare [20], [22], [32], [28](§22):

$$\dot{\theta} = -\frac{1}{3}\theta^2 + 2(\omega^2 - \sigma^2) + g_{i,i} \ , \quad (12a)$$

$$(\omega_{ij})^\bullet = -\frac{2}{3}\theta\omega_{ij} - \sigma_{ik}\omega_{kj} - \omega_{ik}\sigma_{kj} + g_{[i,j]} \ , \quad (12b)$$

$$(\sigma_{ij})^\bullet = -\frac{2}{3}\theta\sigma_{ij} - \sigma_{ik}\sigma_{kj} - \omega_{ik}\omega_{kj} + \frac{2}{3}(\sigma^2 - \omega^2)\delta_{ij} + g_{(i,j)} - \frac{1}{3}g_{k,k}\delta_{ij} \ , \quad (12c)$$

where  $\sigma^2 := \frac{1}{2}\sigma_{ij}\sigma_{ij}$  and  $\omega^2 := \frac{1}{2}\omega_{ij}\omega_{ij}$ .

We can read these equations in the sense that they reconstruct the acceleration gradient  $(g_{i,j})$  in terms of kinematical variables: eq. (12a) gives the trace of  $(g_{i,j})$ , eq.



(12b) its antisymmetric part, and eq. (12c) its tracefree symmetric part, the *Newtonian tidal tensor*  $E_{ij} := g_{i,j} - \frac{1}{3}g_{k,k}\delta_{ij}$ .

We may now establish a system of equations for the variables  $\varrho$ ,  $\theta$ ,  $\sigma_{ij}$  and  $\omega_{ij}$  by using the Euler–Newton system (1). This system only constrains the trace and the antisymmetric part of  $(g_{i,j})$ , but not the tidal tensor  $E_{ij}$ ; we arrive at:

$$\dot{\varrho} = -\varrho\theta \quad , \quad (13a)$$

$$\dot{\theta} = -\frac{1}{3}\theta^2 + 2(\omega^2 - \sigma^2) + \Lambda - 4\pi G\varrho \quad , \quad (13b)$$

$$(\omega_i)^\bullet = -\frac{2}{3}\theta\omega_i + \sigma_{ij}\omega_j \quad , \quad (13c)$$

$$(\sigma_{ij})^\bullet = -\frac{2}{3}\theta\sigma_{ij} - \sigma_{ik}\sigma_{kj} - \omega_{ik}\omega_{kj} + \frac{2}{3}(\sigma^2 - \omega^2)\delta_{ij} + E_{ij} \quad ; \quad (13d)$$

(we have expressed the vorticity tensor in terms of the vector  $\vec{\omega} = \frac{1}{2}\nabla \times \vec{v}$  by means of the formula  $\omega_{ij} = -\epsilon_{ijk}\omega_k$ ).

There have been efforts in the literature to close this system of *ordinary* differential equations by using the corresponding equations of General Relativity (compare [1] and the lecture by Matarrese) and looking at their Newtonian limits. (In fact, the eqs. (13a–d) are formally identical to their GR counterparts in “comoving”, i.e., Lagrangian coordinates.) However, it turns out that, although we can formally derive an evolution equation for the tidal tensor, the system of equations (13) supplemented by the evolution equation for the tidal tensor is not a system of *ordinary* differential equations and the problem remains “non–local” [2], [24], [23], [21].

As a matter of fact, in Newtonian theory we don’t need evolution equations for tracefree symmetric tensors (like the shear and tidal tensors) to get a closed system of equations. We have already obtained such a system without them: the Lagrange–Newton system (8), which is a set of *partial* differential equations.

To see the relation of the system of equations (13) to the Lagrange–Newton system we can show the following (the proof I leave to the reader as an exercise):

i.) Eq. (13a), the *continuity equation*, is equivalent to eq. (1b) and is integrated in the Lagrangian framework by (2e).

ii.) Eq. (13b), *Raychaudhuri’s equation*, is equivalent to eq. (1d), if  $\vec{g}$  is related to the velocity as in eq. (1a).

iii.) Eq. (13c), the *Kelvin–Helmholtz vorticity transport equation*, is equivalent to eq. (1c), if  $\vec{g}$  is related to the velocity as in eq. (1a).

Note that eq. (13c) can also be integrated exactly in the Lagrangian picture, a result due to Cauchy (see [31] – a good textbook on Lagrangian dynamics):

$$\vec{\omega} = \frac{1}{J}\vec{\omega}(\vec{X}, t_0) \cdot \nabla_0 \vec{f} \quad . \quad (14)$$

We arrive at the following conclusions of this section:

- The equations (13a–c) are equivalent to the equations (1b–d) provided the relation between velocity and acceleration is given by (1a).
- The equations (13b,c), if expressed in terms of Lagrangian coordinates, yield a closed Lagrangian system, the Lagrange–Newton system (8a–d).
- The evolution of the tracefree symmetric tensors  $\sigma_{ij}$  and  $E_{ij}$  can be calculated *after* a solution to the Lagrange–Newton system is obtained, the formulas can be read off from eq. (6) (and a similar equation for the velocity gradient):

$$\sigma_{ij} = \frac{1}{2J} \epsilon_{j p q} \mathcal{J}(\dot{f}_i, f_p, f_q) - \frac{1}{6J} \epsilon_{o p q} \mathcal{J}(\dot{f}_o, f_p, f_q) \delta_{ij} \quad ; \quad (15a)$$

$$E_{ij} = \frac{1}{2J} \epsilon_{j p q} \mathcal{J}(\ddot{f}_i, f_p, f_q) - \frac{1}{6J} \epsilon_{o p q} \mathcal{J}(\ddot{f}_o, f_p, f_q) \delta_{ij} \quad (15b)$$

$$= \frac{1}{2J} \epsilon_{j p q} \mathcal{J}(\ddot{f}_i, f_p, f_q) - \frac{1}{3} \left( \Lambda - \frac{4\pi G}{J} \bar{\rho} \right) \delta_{ij} \quad . \quad (15c)$$

We could use eq. (15c) to give another way of stating the Lagrange–Newton system: it is equivalent to the conditions that  $E_{ij}$  is symmetric and tracefree (compare (7)):

$$E_{[i,j]} = 0 \quad \Leftrightarrow \quad (8a, b, c) \quad ;$$

$$E_{ii} = 0 \quad \Leftrightarrow \quad (8d) \quad .$$

- The formulas (14) and (15) are examples of formulas which express some Eulerian field quantity in terms of  $\vec{f}$ . Given any solution for  $\vec{f}$ , we can insert it and use the inverse of the same solution to map the field into Eulerian space as in (9). (Integrals of the Eulerian evolution equations along the trajectories without knowledge of the trajectories themselves do not provide this information directly.) The perturbation solutions discussed in the next section will provide examples.

### 3. Lagrangian Perturbation Theory

The previous sections have equipped us with the necessary framework in which the Lagrangian dynamics of self-gravitating flows can be studied. We have reduced the description of the dynamics of any Eulerian field to the problem of finding the field of trajectories  $\vec{f}$  as a solution of the Lagrange–Newton system (8). This problem will be addressed now.

### 3.1. The Lagrangian perturbation approach

As in the Eulerian case we are not able to write down a solution for generic initial data (i.e. without any symmetry assumptions like plane or spherical symmetry). We may start with the simplest class of solutions, the homogeneous–isotropic ones, and then investigate a perturbative treatment of inhomogeneities. For homogeneous solutions the deformation tensor  $(f_{i|k})$  does not depend on  $\vec{X}$  and the trajectory field is that of a *homogeneous deformation*:

$$f_i^H(\vec{X}, t) = a_{ij}(t)X_j \quad ; \quad a_{ij}(t_0) := \delta_{ij} \quad ; \quad (16a, b)$$

we assume hereafter that it is isotropic,

$$a_{ij}(t) = a(t)\delta_{ij} \quad . \quad (16c)$$

Inserting this ansatz into the Lagrange–Newton system (8) yields the well-known equation

$$\frac{\ddot{a}}{a} = \frac{\Lambda - 4\pi G \varrho_H}{3} \quad , \quad (16d)$$

where  $\varrho_H = \varrho_H(t_0)a^{-3}$  is the homogeneous density field calculated from (16a,c) and (2e). We may use it to integrate eq. (16d) yielding Friedmann’s differential equation:

$$\frac{\dot{a}^2 - e}{a^2} = \frac{8\pi G \varrho_H + \Lambda}{3} \quad ; \quad e = \text{const} \quad . \quad (16e)$$

Solutions of (16e) cover the standard Friedmann cosmologies. We now define the (not necessarily small) deviation  $\vec{p}$  from the “background model”  $\vec{f}^H$ , so that the full displacement map  $\vec{f}$  of an inhomogeneous model reads:

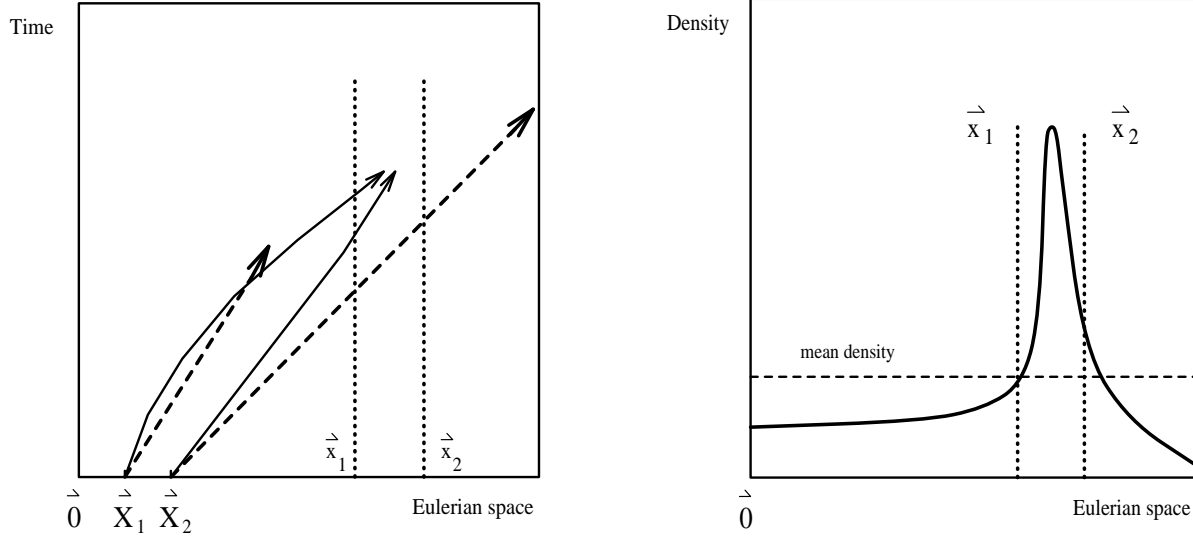
$$\vec{f} = \vec{f}^H + \vec{p}(\vec{X}, t) \quad ; \quad \vec{p}(\vec{X}, t_0) := \vec{0} \quad . \quad (17a, b)$$

(Eq. (17b) expresses the freedom of starting at the same initial position as in the homogeneous case.)

The *Lagrangian perturbation approach* then consists of solving the Lagrange–Newton system for powers of  $\varepsilon$  to obtain the  $m$ th order solution:

$$\vec{f}^{[m]} = \vec{f}^H + \vec{p}^{[m]} \quad ; \quad \vec{p}^{[m]} = \sum_{i=1}^m \varepsilon^i \vec{p}^{(i)} \quad . \quad (17c, d)$$

Note the following important differences to the Eulerian perturbation approach: 1. we need not perturb the density field as in the Eulerian case; 2. the perturbed flow  $\vec{f}$  is expressed in terms of coordinates which are constant along this flow, while in the Eulerian case the perturbation is expressed in a fixed frame, which does not change by moving away from the homogeneous flow. This implies that, although deviations  $\vec{p}$



**Figure 2.** The unperturbed Hubble-flow is represented by the motion of two fluid particles (dashed lines) and the perturbed flow is superimposed (full lines), Fig.2a. In spite of the smallness of the deviations from the Hubble-flow compared to the Hubble-displacements, the density in Eulerian space can experience large changes, Fig.2b.

from  $\vec{f}^H$  might be small compared to the homogeneous displacement, the Eulerian fields evaluated along the perturbed flow can experience large changes (Fig.2).

Solutions of the form (17) can be found and in the following subsections we shall learn more about them.

### 3.2. A comment on average flows

In the ansatz (17) we have splitted the inhomogeneous model into a homogeneous-isotropic model and a deviation from it. Before we discuss solutions of the form (17) we have to look at another assumption which has to be imposed on the inhomogeneous models. This will be necessary for their application to cosmology.

In cosmology we want the homogeneous-isotropic deformation to describe the *mean motion* of the whole universe, and we want to interpret this motion as a reference frame (associated with, e.g., the microwave background). This interpretation requires that the background model (which, so far, is a homogeneous solution) correctly describes the *average flow*, i.e., inhomogeneities or peculiar-motions must vanish on average on some large scale. This we call the *cosmological principle of homogeneity*. That the average flow is isotropic is another assumption which we set *a priori*. It is, however, supported by the extreme isotropy of the microwave background. Therefore, in order to make Lagrangian perturbation models applicable to reality, we have to carefully analyze the average of inhomogeneous models. For this purpose we may look at the

general expansion law in Newtonian cosmology by averaging the system of equations (13). A general discussion is to be found in [15]. This study shows that the general expansion law for an expansion function  $a_{\mathcal{D}} := V^{1/3}$ , where  $V$  is the volume of the spatial domain  $\mathcal{D}(t)$  on which the average is performed, is of the form:

$$3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} + 4\pi G \frac{M_{\mathcal{D}}}{a_{\mathcal{D}}^3} - \Lambda = -\frac{2}{3} \left( a_{\mathcal{D}}^{-3} \int_{\mathcal{D}(t)} \nabla \cdot \vec{u} \right)^2 + a_{\mathcal{D}}^{-3} \int_{\mathcal{D}(t)} \nabla \cdot \vec{\mathcal{Q}} \ , \quad \vec{\mathcal{Q}} := (\vec{u} \nabla \cdot \vec{u} - \vec{u} \cdot \nabla \vec{u}) \ , \quad (18a)$$

where  $M_{\mathcal{D}}$  is the total mass contained in  $\mathcal{D}(t)$ . Eq. (18a) shows that the average expansion is of Friedmann type,  $a_{\mathcal{D}} \equiv a$ , only in the case where the averages of the nonlinear terms on the r.h.s. (which depend on the inhomogeneities) vanish (compare (16d)). In other words, if we follow inhomogeneities into the nonlinear regime (and that's what we have in mind with Lagrangian approximations), then we actually neglect the influence of the inhomogeneities on the global expansion when we use homogeneous solutions for the average flow. Since these nonlinear source terms can be written as a divergence [15], we can use Gauß's theorem to transform the volume integrals in (18a) into surface integrals over the boundary of the domain  $\partial\mathcal{D}$ , e.g.:

$$\int_{\mathcal{D}(t)} d^3x \nabla \cdot \vec{\mathcal{Q}} = \int_{\partial\mathcal{D}(t)} d\vec{S} \cdot \vec{\mathcal{Q}} \ . \quad (18b)$$

The flux of  $\vec{u}$  and  $\vec{\mathcal{Q}}$  through the boundary of the averaging volume is assumed to be negligible. This can only be true, if the extent of structures in the Universe is considerably smaller than the averaging domain. In standard cosmologies we assume that this is indeed the case on some large scale where we apply our models, and we even assume that this boundary flux is exactly zero.

It is interesting to see at this stage that we implicitly satisfy this assumption by imposing *periodic boundary conditions* on the cosmic peculiar-fields: Periodicity means that we can alternatively view the Universe as a torus. A torus is a compact space without boundary, and consequently any flux through the boundary vanishes exactly.

### 3.3. Perturbation solutions at first order

In this subsection we shall write down the *general* first-order solution in order to see that we need quite a list of arguments to reduce it to what we call the “Zel’dovich–approximation” [34], [35].

A solution to the basic system of equations (8) is *general*, if we can specify initial data for the velocity *independently* of a given density field (or acceleration field, respectively). If we split these data into their homogeneous parts (belonging to the background solution) and deviations from the background, *peculiar-velocity*  $\vec{u}$

and *peculiar-acceleration*  $\vec{w}$ , then we have to specify

$$\vec{U} := \vec{u}(\vec{X}, t_0) =: \vec{U}^L + \vec{U}^T ; \quad \vec{W} := \vec{w}(\vec{X}, t_0) , \quad (19a)$$

where we have splitted the peculiar-velocity into its irrotational (longitudinal) part  $\vec{U}^L$  and its rotational (transverse) part  $\vec{U}^T$  (the acceleration is irrotational in Newton's theory). We may introduce scalar potentials and a vector potential such that

$$\vec{U}^L =: \nabla_0 S ; \quad \vec{U}^T =: \nabla_0 \times \vec{S} ; \quad \vec{W} =: \nabla_0 \Phi . \quad (19b)$$

Solving the Lagrange-Newton system to the first order with the perturbation ansatz (17) we obtain [9]:

$$\vec{f}^{[1]} = a(t)\vec{X} + b_1(t)\nabla_0 S^{(1)}(\vec{X}) + b_2(t)\nabla_0 \times \vec{S}^{(1)}(\vec{X}) + b_3(t)\nabla_0 \Phi^{(1)}(\vec{X}) , \quad (20)$$

where the  $b_i(t)$  depend on the chosen background solution  $a(t)$ ,  $b_i(t_0) = 0$ , and the  $\vec{X}$ -dependent perturbation potentials have to obey Poisson equations which relate them to the initial data (19b):

$$\Delta_0 S^{(1)} = \Delta_0 S t_0 ; \quad \Delta_0 \vec{S}^{(1)} = \Delta_0 \vec{S} t_0 ; \quad \Delta_0 \Phi^{(1)} = \Delta_0 \Phi t_0^2 . \quad (20a - e)$$

( $\Delta_0$  denotes the Laplace operator with respect to Lagrangian coordinates.)

The solution (20) shows that the evolution of a fluid element depends (non-locally) on the distribution of all fluid elements, because Poisson equations have to be solved (which is clear in view of the structure of the theory). The rotational part stems from the Lagrangian equations (8a-c) which must not be forgotten! Only at first order the rotational part directly represents rotational flow in Eulerian space. At higher orders rotational parts arise also for irrotational flow. This can be explained by the following consideration: Imagine an arrow for which we require that it keeps a fixed angle to some axis in an Eulerian coordinate system. The arrow may be placed on a system which is orbiting around some point. In this (Lagrangian) system the arrow performs a rotating movement, while in Eulerian space it is just translationally shifted. Indeed, if we restrict the motion to be irrotational in Lagrangian space, then this restricts possible classes of motion far more than the requirement of irrotational motion in Eulerian space.

Let us return to the solution (20). At first order it is straightforward to solve the 5 Poisson equations (20a-e) explicitly:

$$S^{(1)} = S t_0 + \Psi ; \quad \vec{S}^{(1)} = \vec{S} t_0 + \vec{\Psi} ; \quad \Phi^{(1)} = \Phi t_0^2 + \Omega ,$$

$$\Delta_0 \Psi = 0 ; \quad \Delta_0 \vec{\Psi} = \vec{0} ; \quad \Delta_0 \Omega = 0 . \quad (21a - e)$$

We can get rid of the harmonic functions  $\Psi$ ,  $\vec{\Psi}$  and  $\Omega$  and we can make  $S$ ,  $\vec{S}$  and  $\Phi$  unique, if we impose periodic boundary conditions on the initial data: the only periodic

harmonic functions are constant in space, a constant which expresses a translation of the whole simulation box and which can be set to zero, since the basic equations are translationally invariant (see [12], Appendix C and [21]). If we don't do that then the solutions are not unique! Removing this freedom reduces (20) to a *local* approximation which is only possible at first order.

For the purpose of simulating the evolution of large-scale structure forward in time we can simplify the solution further by assuming that, initially

$$\vec{U}(\vec{X}) = \vec{W}(\vec{X})t_0 \quad , \quad (22)$$

i.e., peculiar-velocity and -acceleration are parallel to start with (in particular, we have  $\vec{U}^T(\vec{X}) = \vec{0}$ ). This assumption is justified on physical grounds: as long as the perturbations are small, we may describe them by the (Eulerian) linear theory. Then, asymptotically (after the decaying modes have died out) the condition (22) is approximately reached and we may start with this condition from the outset. Employing (22) the first-order solution reduces to the “Zel’dovich-approximation” (compare [9] for the Einstein-de Sitter background, and [8], [4] for general backgrounds including a cosmological constant to see that the time-coefficients come out right):

$$\vec{f}^{[Z]} = a(t)\vec{X} + b(t)\nabla_0 S(\vec{X}) \quad . \quad (23)$$

Note that, strictly speaking, (23) is not the growing part of the general solution, but is that part which results from the restriction (22) on initial data: this restriction automatically cancels the decaying modes at first order, but not at higher orders.

### 3.4. Higher-order solutions

The solution of the Lagrange-Newton system for higher orders in  $\varepsilon$  for the initial data (22) systematically gives us higher-order corrections to the “Zel’dovich-approximation”. For *general* initial data, already the second-order solution needs four pages to show it (compare [11] for the longitudinal part, which comprises the solution of one of the four equations (8a-d)). Restricting them to the setting (22) we obtain the much simpler form which is commonly being studied ([5], compare Bouchet’s lecture, and [10], [11] for alternative models of this form):

$$\vec{f}^{[2]} = a(t)\vec{X} + q_1(t)\nabla_0 \mathcal{S}^{(1)}(\vec{X}) + q_2(t)\nabla_0 \mathcal{S}^{(2)}(\vec{X}) \quad . \quad (24)$$

The perturbation potentials in (24) have to be constructed by solving iteratively the two Poisson equations:

$$\Delta_0 S^{(1)} = I(S_{|i|k})t_0 \quad , \quad (24a)$$

$$\Delta_0 S^{(2)} = 2II(S_{|i|k}^{(1)}) \quad . \quad (24b)$$

$I$  and  $II$  denote the first and second principal scalar invariants of the tensor gradient in brackets:

$$I(S_{|i|k}) = \text{tr}(S_{|i|k}) = \Delta_0 S \quad , \quad (24c)$$

$$II(S_{|i|k}^{(1)}) = \frac{1}{2}[(\text{tr}(S_{|i|k}^{(1)}))^2 - \text{tr}((S_{|i|k}^{(1)})^2)] \quad . \quad (24d)$$

The third-order solution for the setting (22) may be found in [12], the time-coefficients being evaluated for an Einstein-de Sitter background. (For other backgrounds we cannot write them in explicit form, see [18] for a thorough discussion.) Illustrations are given in Figs. 3,4.

### 3.5. Comparison of Eulerian and Lagrangian perturbation theory

For the comparison of both perturbation approaches we look at the equations governing the evolution of inhomogeneities in the first-order Eulerian and Lagrangian approximations. Consider the contrast density  $\Delta := (\varrho - \varrho_H)/\varrho$ ,  $-\infty < \Delta < 1$ , which is more adapted to the nonlinear situation than the conventional definition  $\delta = (\varrho - \varrho_H)/\varrho_H = \Delta/(1 - \Delta)$ , defined in Eulerian perturbation theory. For this field we can find the following exact evolution equations by inserting the continuity equation into Raychaudhuri's equation (13b) and splitting off the homogeneous part [29], [8], [9]:

$$\dot{\Delta} = (\Delta - 1)I \quad , \quad (25a)$$

$$\ddot{\Delta} + 2\frac{\dot{a}}{a}\dot{\Delta} - 4\pi G\varrho_H\Delta = (\Delta - 1)2II \quad , \quad (25b)$$

where  $I$  and  $II$  denote here the invariants of the peculiar-velocity tensor gradient with respect to Eulerian coordinates which are scaled by the expansion factor,  $q_i = x_i/a(t)$ :  $(\partial u_i/\partial q_j)$ .

For  $II = 0$  the equations (25) are (except for the Lagrangian time derivative and the redefinition of the density contrast) the same equations as known in Eulerian linear theory for  $\delta(\vec{q}, t)$  [28]:

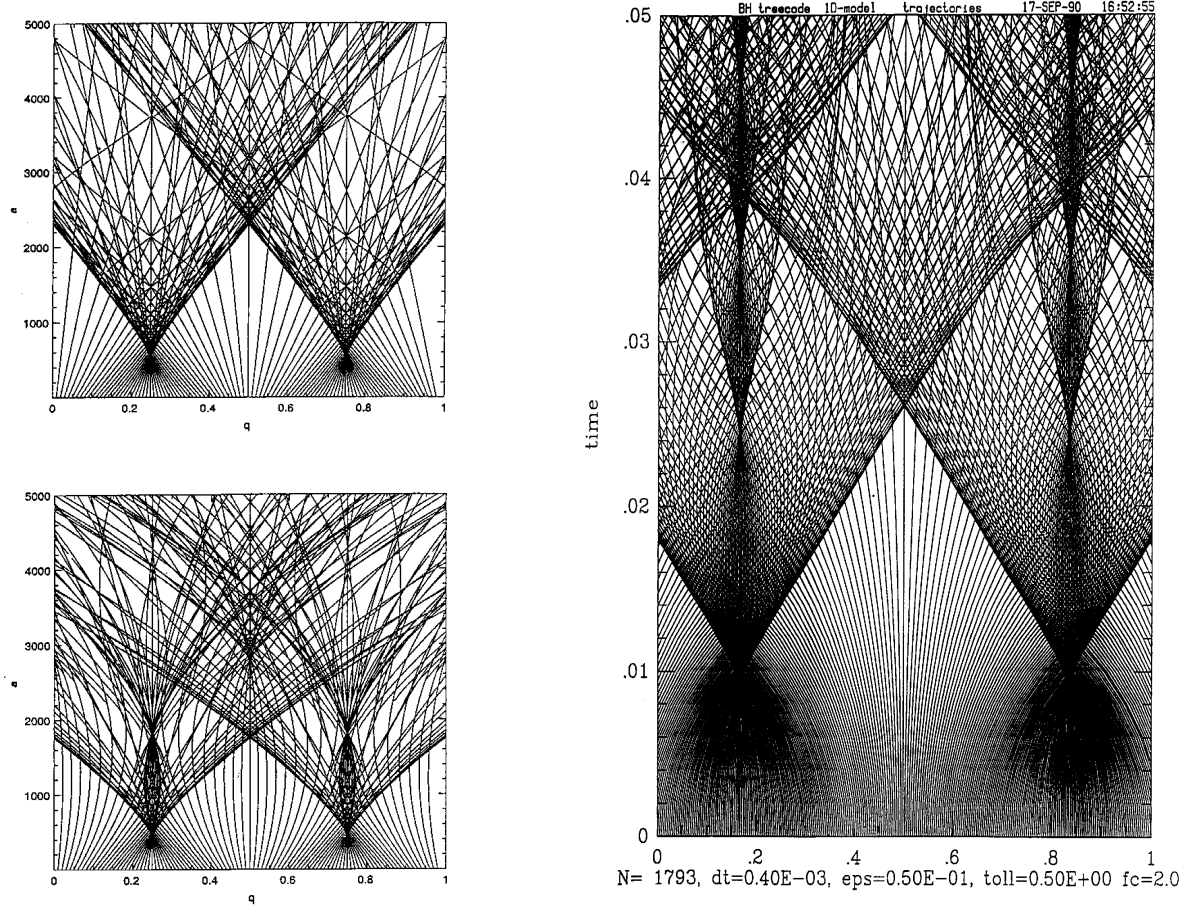
$$\frac{\partial}{\partial t} \Big|_q \delta^\ell = -I^\ell \quad , \quad (25a)^\ell$$

$$\frac{\partial^2}{\partial t^2} \Big|_q \delta^\ell + 2\frac{\dot{a}}{a}\frac{\partial}{\partial t} \Big|_q \delta^\ell - 4\pi G\varrho_H\delta^\ell = 0 \quad . \quad (25b)^\ell$$

These equations can be obtained by linearizing (25). On the contrary, the Lagrangian linear solution solves the exact equations (25) for  $II = 0$ .

I emphasize that the similarity between the linear and the nonlinear case restricted to  $II = 0$  is nontrivial: a nice exercise is to compute the equation for the conventional



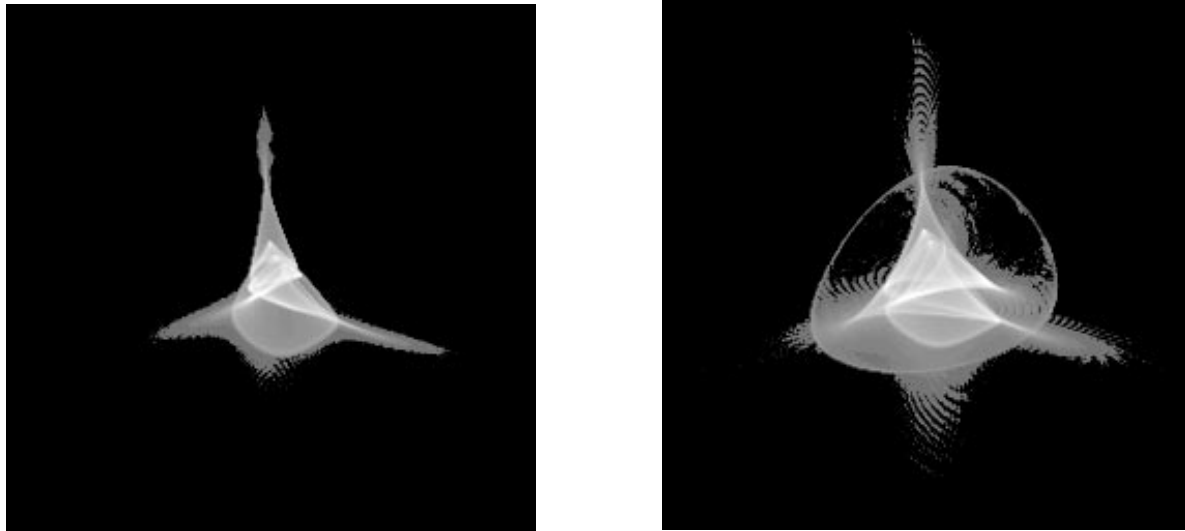


**Figure 3.** The trajectories for a simple plane wave model (2D) are plotted for the first-order (left, top panel) and for the second-order solution (left, bottom panel) (taken from [11]) compared with the trajectories in a two-dimensional tree-code simulation (taken from [17]). Those trajectories are plotted which end in an Eulerian spacetime section to show that the second-order model displays a secondary shell-crossing (this is not visible for trajectories which lie in this plane for all times – they follow a plane-symmetric collapse for which the first-order model is the exact solution). Two generations of caustics successively appear, whereas the numerical simulation shows further generations.

density contrast  $\delta$  from the nonlinear equation (25b) for  $II = 0$  (solved by the Lagrangian first-order solution) using the definitions  $\Delta := \delta/(1+\delta)$  and  $\dot{\cdot} := \partial_t|_q + \frac{1}{a}\vec{u} \cdot \nabla_q$ , and to compare with the Eulerian linear equation (25b)<sup>ℓ</sup>:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\rho_H\delta +$$

$$\ddot{\delta}\delta - 2\dot{\delta}^2 + 2\frac{\dot{a}}{a}\dot{\delta}\delta - 4\pi G\rho_H\delta^2 = 0 \quad . \quad (26)$$



**Figure 4.** A slice through the density fields ( $512^3$  trajectories collected into a  $256^3$  pixel grid) predicted by the Lagrangian approximations at first order (left) and third order (right panel) for a 3D plane wave model is shown (taken from [17]). The second- and third-order schemes feature secondary shell-crossings. This hierarchy of nested caustics continues down to smaller scales as was shown in a two-dimensional numerical simulation in [19].

Hence, first-order Lagrangian perturbations involve nonlinearities due to nonlinearities in the dependent variable  $\delta$ , but also due to products of  $\vec{u}$  and  $\delta$  arising from the convective derivative (contained in the overdot).

This exercise demonstrates the inherently nonlinear character of a Lagrangian perturbation approach. Essentially, this property can be traced back to the implicit solution of the Eulerian convection of the flow  $(\vec{u} \cdot \nabla_q)\vec{u}$  by the Lagrangian time-derivative, and to the fact that the density is exactly known for any solution.

The second-order Lagrangian theory takes the term  $II$  in equation (25) approximately into account, and thus covers essential effects of the tidal action on the fluid.

### 3.6. *Limitations of the Lagrangian perturbation theory*

In order to recognize the limitations of the Lagrangian perturbation approach, two classes of arguments can be given: the first is analytical, based on the approach itself; the second is furnished by comparisons with N-body simulations. Since the latter is the topic of another lecture (by Peter Coles, and ref. therein), I here concentrate on the former, and only give two illustrations of the latter.

Strictly speaking, the Lagrangian perturbation approach is limited by construction

to the regime  $t < t_c$ , where

$$|p_{i,j}(t_c)| \ll a(t_c) . \quad (27)$$

This condition is very conservative in view of the success of these approximations if followed up to shell-crossing, i.e.,  $|p_{i,j}(t_c)| = \mathcal{O}(a(t_c))$  and beyond (see Coles' lecture). The condition (27) has its counterpart in the condition  $\delta \ll 1$  in Eulerian perturbation theory. At the epoch of shell-crossing, all orders of the perturbations yield displacements which are of comparable magnitude; consequently the perturbation approach breaks down there (we only expect it to converge before shell-crossing). In practice, however, we want to follow the solutions beyond shell-crossing (the Lagrange–Newton system is still regular in this regime, i.e., the individual trajectories don't feel their intersection with others). This demands that we must argue whether and how this extrapolation is justified. The examples given in the previous subsection suggest that the extrapolation can indeed be meaningful.

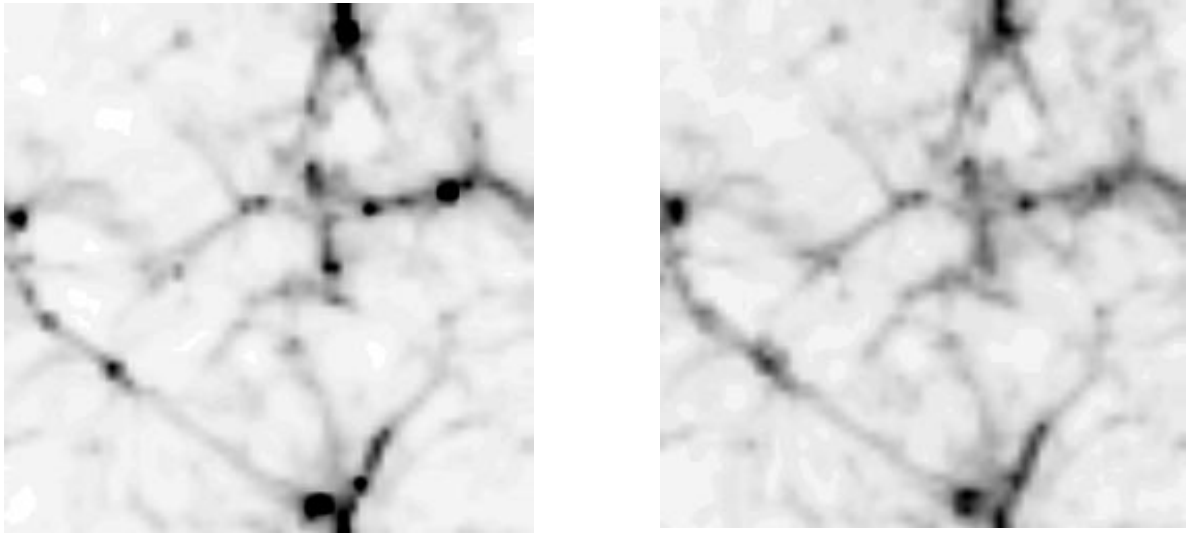
As mentioned earlier, in the regime after shell-crossing  $J < 0$ , the mapping  $\vec{f}$  is no longer unique and the inverse  $\vec{h}$  is multivalued. If we assume that the density can be written as a superposition of the partial densities corresponding to the different streams, then a reasonable extension into the multi-stream regime for the density is

$$\varrho[\vec{x}, t] = \sum_{\gamma=1}^n \varrho(\vec{h}_{\gamma}(\vec{x}, t), t) = \sum_{\gamma=1}^n \frac{\varrho(\vec{h}_{\gamma}(\vec{x}, t))}{|\det(f_{i,k}(\vec{h}_{\gamma}(\vec{x}, t), t))|} ; \quad (28)$$

the total density in the  $n$ -stream region is the sum of the moduli of the individual densities of the streams. This extension is implicitly used when we construct the density field from displaced particles: the formula (28) is approximated by the collection of particles in an Eulerian grid at times after shell-crossing. However, due to the presence of self-gravity the streams interact and the formula (28) has to be taken with caution. Indeed, expressing the superposition assumption in terms of the accelerations we encounter a problem: In the multi-stream region there are  $n$  fluid elements at the same Eulerian position, thus the total gravitational field strength at this position would have to be added up similar to (28) and the fluid particle should be pushed by this total strength. On the contrary, the Lagrangian solutions still push the fluid elements with their individual accelerations. In other words, Einstein's equivalence principle of inertial and gravitational mass is violated: the particles are not accelerated by the total gravitational field strength. We have to find extensions of the theory presented (e.g., in the framework of the Vlasov–Poisson system) to properly deal with this problem.

The performance of the Lagrangian perturbation solutions after shell-crossing as inferred from a number of comparisons with N-body simulations appears to be weak due to the shortcomings discussed above. This is most easily visible in pancake models [13]. If we consider hierarchical cosmogonies, especially models which involve much

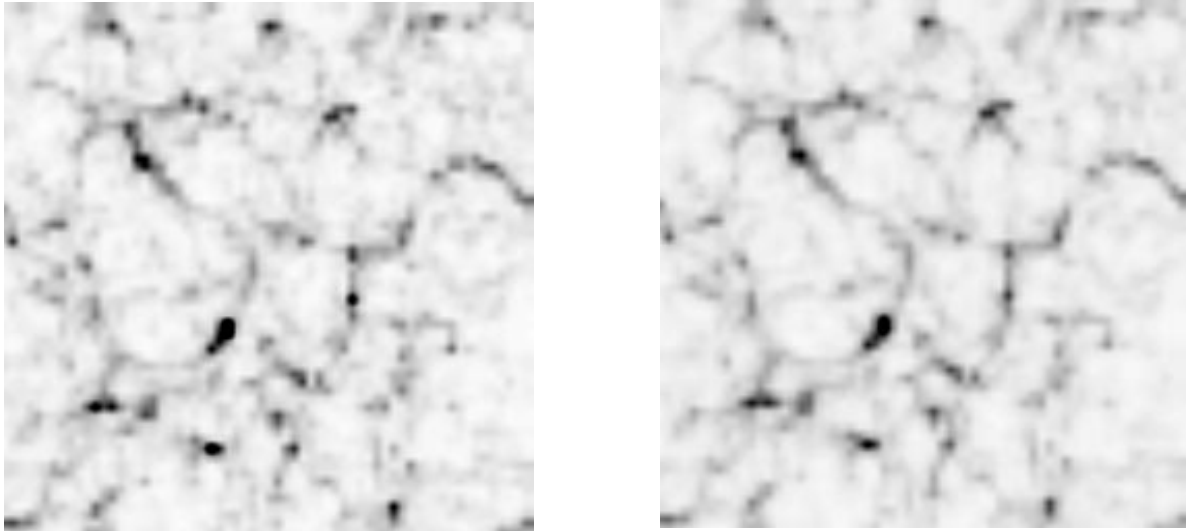
small-scale power, then the power spectrum of fluctuations must be sufficiently steep to avoid shell-crossing on small scales. Otherwise, these high-frequency components have to be truncated or filtered (see Coles lecture and ref. therein). In the former case, a spectral index  $n = -3$  in a power law spectrum  $\propto |\vec{k}|^n$  lies close to the limit between “pancake-type” and “hierarchical” cosmogonies having  $n > -3$ . This is so, because in the case  $n = -3$  the integrated power per wave number interval is approximately constant (logarithmic), so that all waves enter the nonlinear regime at about the same time (Fig.5). For more small-scale power the smoothing scale of high-frequency modes has to be close to (but smaller than) the nonlinearity scale. In Fig.6, I present the result of a comparison for a variant of the CDM cosmogony, the BSI model (see Gottlöber’s lecture, this volume) with less small-scale power than standard CDM.



**Figure 5.** A slice of the density field is shown for a PM simulation (left) and the second-order Lagrangian approximation (right panel) (taken from [27]). The spectrum of fluctuations was given by a powerlaw with index  $n = -3$ , which can be taken to discriminate between the “pancake” and “hierarchical” regimes of structure formation.

Let us again summarize the main conclusions of this section:

- In a Lagrangian perturbation approach only one variable, the trajectory field  $\vec{f}$ , is perturbed. The perturbation is evaluated along the perturbed orbits. Although the perturbations of the orbits might be small, the Eulerian fields evaluated along the perturbed orbits can experience large changes.
- The first-order Lagrangian approximation reduces to the “Zel’dovich-approximation”, if the initial data are restricted such that peculiar-velocity and peculiar-acceleration are parallel.



**Figure 6.** A slice of the density field in a  $200h^{-1}\text{Mpc}$  box is shown for a PM simulation (left) and the second-order Lagrangian approximation (right panel) (taken from [33]). The spectrum of fluctuations was a (COBE normalized) non-standard CDM spectrum with broken scale invariance (BSI). The spectrum had to be smoothed at the high-frequency end for the Lagrangian scheme to avoid substantial post-singularity evolution.

- Imposing periodic boundary conditions on the cosmic peculiar-fields is necessary to guarantee 1. uniqueness of the solutions, and 2., to assure that the average flow is given by the homogeneous “background” solutions.
- Higher-order effects amount to additional internal structures such as “second generation” pancakes, filaments and clusters. They also accelerate the collapse.
- A Lagrangian perturbation solution inherently includes nonlinear terms. This is also true for the first-order solution, which is linear in Lagrangian space.
- The temporal limit of application of the Lagrangian schemes is reached at the epoch of shell-crossing, where the displacements for each order attain similar magnitudes. Following these schemes beyond shell-crossing time requires truncation of high-frequency modes in the initial fluctuation spectrum, thus defining a spatial lower limit of application, which roughly corresponds to galaxy group mass scales,  $10^{13}M_{\odot}$  [26], [16].

The Lagrangian perturbation approach is investigated to arbitrary order in [21]. In that paper as well as in the review articles [3], [6], [14] and [30] you may find references to related work.

**Acknowledgments:** This work considerably gained from many discussions with Jürgen Ehlers. I wish to thank him and H. Wagner for helpful comments on the manuscript. I acknowledge financial support by the “Sonderforschungsbereich 375–95 für Astro–Teilchenphysik” der Deutschen Forschungsgemeinschaft.

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